2 Sets, Functions, Sequences, and Sums

2.3 Functions

1. a function f from A to B is an assignement of a unique value of B to each value of A. (note that this means each value of A can only be mapped to a unique value, and also, each value of A has to be mapped to some value of B.)

2. the set A above is called the domain, and the set B is called the codomain. A subset of the codomain makes the range of f, and that subset is the set of particular values of B that get assigned to values of A.

3. Let $a \in A$ and say that $f(a) = b$, of course with $b \in B$. Then b is called the image of a, and a is called the preimage of b. Then f is said to map a to b. The set of all values b will make up the range of f, as defined above.

4. two functions f and g are equal if they have the same domain and codomain, and $f(x) = g(x)$ for every value x of the domain

5. two functions can be added, subtracted, divided and multiply if they have the same domain (so that the new function will be defined)

6. a function is strictly increasing iff: $\forall x, y, \Big((x < y) \to (f(x) < f(y))\Big)$.

7. a function is increasing iff: $\forall x, y, \Big((x < y) \to (f(x) \leq f(y))\Big)$.

8. a function is strictly decreasing iff: $\forall x, y, \Big((x < y) \to (f(x) > f(y))\Big)$.

9. a function is decreasing iff: $\forall x, y, \Big((x < y) \to (f(x) \geq f(y))\Big)$.

10. a function is one-to-one or injective iff (that is if and only if):

$$\forall x, y, \Big((f(x) = f(y)) \to (x = y)\Big)$$

For example, the function $f(x) = 2x + 3, f : \mathbb{R} \to \mathbb{R}$ is one-to-one, but the function $f(x) = 2x^2 + 3, f : \mathbb{R} \to \mathbb{R}$ is not (prove them to convince yourself).

11. a function is onto or surjective iff:

$$\forall y \in B, \exists x \in A \ (f(x) = y)$$

For example, the function $f(x) = 2x + 3, f : \mathbb{R} \to \mathbb{R}$ is onto, but the function $f(x) = 2x^2 + 3, f : \mathbb{R} \to \mathbb{R}$ is not (prove them to convince yourself).

12. a function that is both one-to-one and onto is a one-to-one correspondence or bijective. All linear functions are bijectives from reals to the reals ($f : \mathbb{R} \to \mathbb{R}$)

13. if a function $f : A \to B$ is bijective (or one-to-one correspondence) with $f(x) = y$ then there is an inverse function $f^{-1} : B \to A$ with $f(y) = x$. For example, for $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$, the inverse function is $f^{-1} : \mathbb{R} \to \mathbb{R}$ defined by $f^{-1}(x) = \frac{x-1}{2}$ (note that the inverse function is not defined by $f(x) = \frac{1}{2x+1}$)

14. a one-to-one correspondence f is called invertible because there is a function (namely f^{-1}) that is the inverse of f

15. the composition of two functions f and g is defined by $(f \circ g)(x) = f(g(x))$. For example, for $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 2$ and $g(x) = x^2 - 3$, then $(f \circ g) : \mathbb{R} \to \mathbb{R}$, $(f \circ g)(x) = f(g(x)) = f(x^2 - 3) = 3(x^2 - 3) - 2 = 3x^2 - 11$, however, $(g \circ f) : \mathbb{R} \to \mathbb{R}$, $(g \circ f)(x) = g(f(x)) = g(3x - 2) = (3x - 2)^2 - 3 = 9x^2 - 12x + 1$

16. note that $f \circ f^{-1} = f^{-1} \circ f = id$, where id is the identity function that takes any value to itself ($id(x) = x$ for any domain and codomain). And so $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$, for all x values of the domain

17. the inverse image of a set $\{y\}$ in the Range is $f^{-1}(\{y\}) = a$ iff $f(a) = y$ (so it is the value that got mapped to y). For the inverse image of a set, the function does not have to be bijective, and so the inverse image could be more than one value. Example: let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Then $f^{-1}(4) = \{-2, 2\}$ and $f^{-1}(\{4, 5\}) = \{-\sqrt{5}, -2, 2, \sqrt{5}\}$. Note that the inverse image is defined for a set, not for a value (and so the set could have just one element, as shown above with $S = \{4\}$)–definition on page 147

18. the graph of the function is the set of ordered pairs $(x, f(x))$ for all x values in the domain

19. the floor function $\lfloor x \rfloor : \mathbb{R} \to \mathbb{R}$ is the largest integer that is less than or equal to x (Example $\lfloor 3.87 \rfloor = 3$ and $\lfloor -3.87 \rfloor = -4$

20. the ceiling function $\lceil x \rceil : \mathbb{R} \to \mathbb{R}$ is the smallest integer that is greater than or equal to x (Example $\lceil 3.27 \rceil = 4$ and $\lfloor -3.87 \rfloor = -3$

21. properties of floor and ceiling functions page 144 (note that n is an integer, but x can be any real number)

22. the factorial function $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$. For example $f(4) = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$